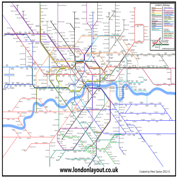
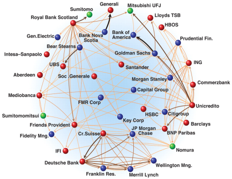
Centralities, perturbations, resilience in network models.

Fabio Fagnani,
DISMA Department of Mathematical Sciences
Politecnico di Torino

Kick off meeting Project of Excellence

The behavior of complex infrastructures





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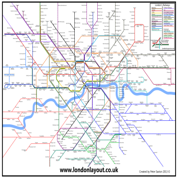

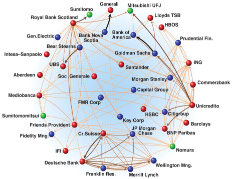

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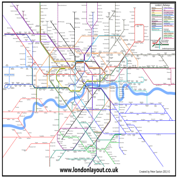

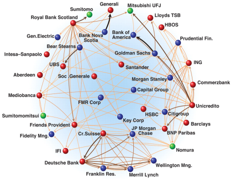

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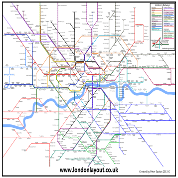
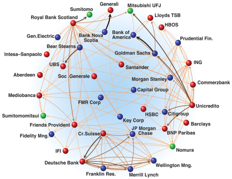
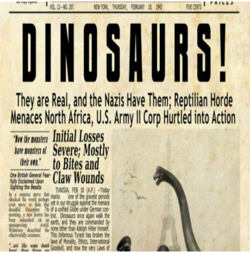




- Designed to work under 'normal' operative conditions.
- Local failures, exhogeneous events.
- Systemic risk: spreading and amplification of the perturbation. The domino effect
- Resilience: the capacity of a system to absorb the effect of a perturbation.

...and of social networks


VOL. 12—NO. 207. NEW YORK, THURSDAY, FEBRUARY 18, 1943 FIVE CENTS

DINOSAURS!

They are Real, and the Nazis Have Them; Reptilian Horde Menaces North Africa, U.S. Army II Corp Hurtled into Action

"Now the monsters have monsters of their own."

One British General Fearfully Exclaimed Upon Sighting the Beasts

In a surprise move that shocked the world perhaps even more so than that dreadful December 7th morning, a new horror has been unleashed on an unsuspecting world. Witnesses described the otherworldly creatures.

"...not like some dumb

Initial Losses Severe; Mostly to Bites and Claw Wounds

TUNISIA, FEB 18 (A.P.) --Today marks one of the gravest periods yet in our struggle against the menace of a unified Globe under German control. Dinosaurs once again walk the earth, and they are commanded by none other than Adolph Hitler himself. This Infamous Tyrant has broken the laws of Morality, Ethics, International Goodwill, and now the very Laws of


WEEKLY WORLD NEWS

THE WORLD'S ONLY RELIABLE NEWSPAPER

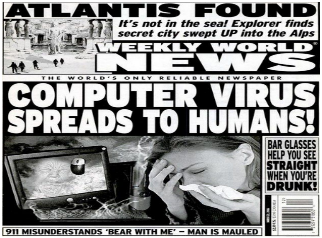
DOG LIVES INSIDE A PLASTIC BUBBLE

WORLD'S SADDEST BOOK CAUSES DOZENS OF SUICIDES!

HOLY MACKEREL! Village worships sacred fish

MOON TO EXPLODE IN 6 MONTHS!

U.S. GOV'T COVERING UP SHOCKING TRUTH!


ATLANTIS FOUND

It's not in the sea! Explorer finds secret city swept UP into the Alps

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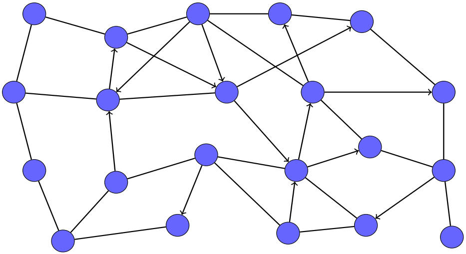
911 MISUNDERSTANDS 'BEAR WITH ME' – MAN IS MAULED

The general picture

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ graph



An 'object' attached to $\mathcal{G}$:

- A static vector $\pi \in \mathbb{R}^{\mathcal{V}}$
- a dynamical system $\pi(t)$.

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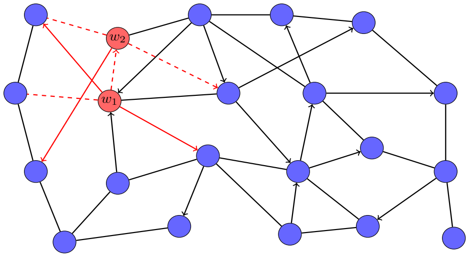

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Study the effect on the 'object' of perturbations on $\mathcal{G}$:

- The effect of a (local) rewiring. Optimization issues.
- Resilience to small/local perturbation.
- The large scale limit $n = |\mathcal{V}| \to +\infty$.

What is classical

- The effect of perturbations on graph connectivity.
- Results for families of random graphs.
- Percolation.

An example of ongoing research: Bonacich centrality

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Generalization: any weight matrix W.

The computation of π

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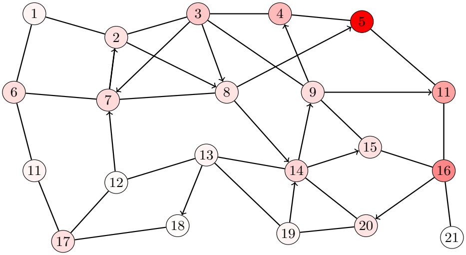

The dynamics beyond π

Two dynamical systems connected to P:

- $x_i(t+1) = \sum_j P_{ij} x_j(t)$ *Averaging dynamics*
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Some key issues:

- Behavior in large scale graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $n = |\mathcal{V}| \to +\infty$
  - infer properties of π without explicit computation ($||\pi||_\infty \to 0$)
- Network engineering problems:
  - shaping π by local rewiring
  - centrality optimization
- Fundamental limitations to the effect of local perturbations. Resilience properties.

A general optimization problem

$\mathcal{G} = (\mathcal{V}, \mathcal{E}) \to P$

$\pi = P'\pi$ centrality

$\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}}) \to \tilde{P}$

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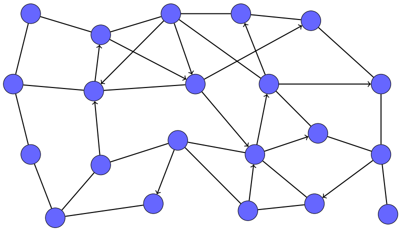

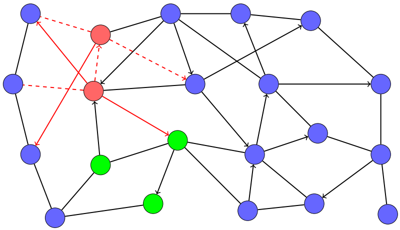

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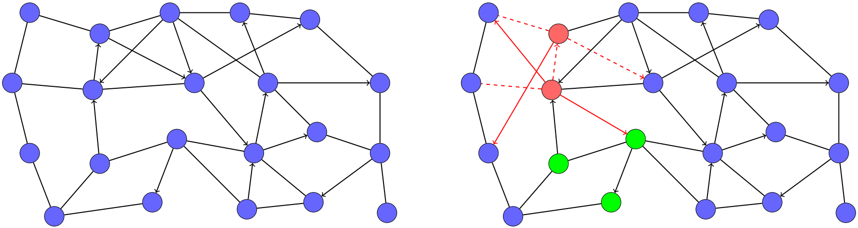
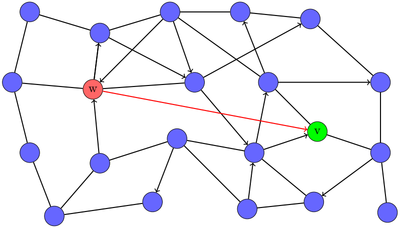


$\mathbf{G}_{\mathcal{G}}$ family of admissible perturbations of $\mathcal{G}$, $\mathcal{D} \subseteq \mathcal{V}$ target subset.

Problem: $\underset{\tilde{\mathcal{G}} \in \mathbf{G}_{\mathcal{G}}}{\operatorname{argmax}} \, \tilde{\pi}(\mathcal{D})$

The effect of adding an edge



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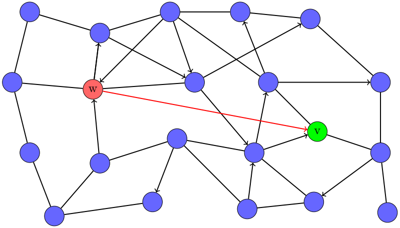

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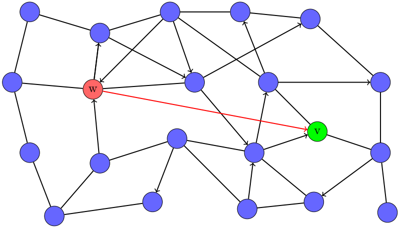

- $\tilde{\pi}_v > \pi_v$
- $\underset{v \in \mathcal{V}}{\text{argmax}}\, \tilde{\pi}_w = \underset{v:(v,w)\in\mathcal{E}}{\text{argmax}}\, Z_{vw}$
- $Z_{ij} := \sum_{t=0}^{+\infty} [P_{ij}^t - \pi_j]$
  Fundamental matrix

Fundamental limitations

P and $\tilde{P}$ irreducible stochastic matrices on $\mathcal{V}$

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The result we are looking for:

Small perturbation in a large network has a small effect:

$|\mathcal{W}| << n = |\mathcal{V}| \;\Rightarrow\; \pi - \tilde{\pi} \to 0$

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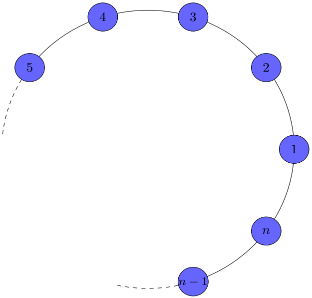
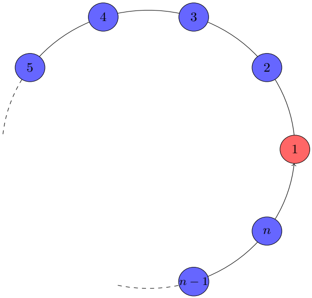
- $||\tilde{P} - P||_p \geq \max\{|\tilde{P}_{ij} - P_{ij}|\}$ bounded away from 0 independently on the size n.

Example 1





$P_{ii+1} = P_{ii-1} = 1/2,$

$\pi = P'\pi$ uniform

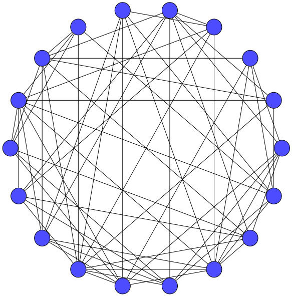
$\tilde{P}_{1,2} = 1, \ \tilde{P}_{1,n} = 0$

$\tilde{\pi}_1 = 1/n, \ \tilde{\pi}_j = \frac{2(n-j+1)}{n^2} \ j \geq 2$

$$||\pi - \tilde{\pi}||_1 \asymp \text{ Const.}$$

Example 2

$\mathcal{G} = ER(n,p)$ Erdos-Renyi random graph with n nodes.



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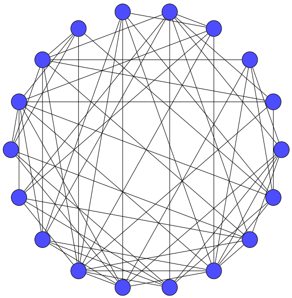
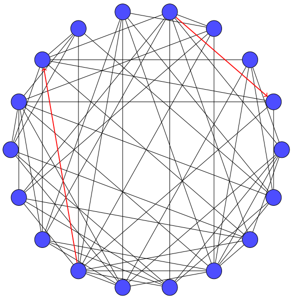


$\tilde{\mathcal{G}}$ is obtained from $\mathcal{G}$ erasing two edges



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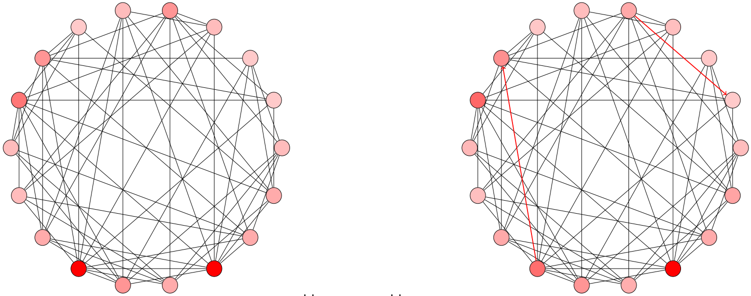

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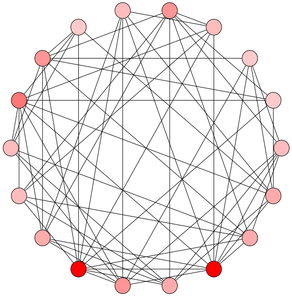

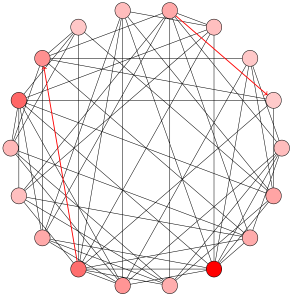

$$||\tilde{\pi} - \pi||_1 = 0.03$$

Well connected graphs $\Rightarrow$ Fast mixing $\Rightarrow$ Resilience

Page rank centrality

$\mathcal{G}$ web graph.

Page-rank centrality: $\pi_i^{pr} = \alpha\mu_i + (1-\alpha)\sum_j Pji\pi_j^{pr}$

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$$||\tilde{\pi}^{pr} - \pi^{pr}||_1 \leq \frac{8}{\log(1-\alpha)}\pi^{pr}(\mathcal{W})$$

Any modification of the hyperlinks from a set $\mathcal{W}$ of webpages, generates a perturbation of the page-rank centrality whose 1-norm is bounded by the original page-rank centrality of $\mathcal{W}$.

Production network

$\mathcal{G}$ graph of production interactions. P_{ij} fraction of the goods used by firm i in its production coming from firm j.

Profits: $\pi_i^{pr} = \alpha\mu_i + (1-\alpha)\sum_j Pji\pi_j^{pr}$

Perturbations: price distortions, changes in the production topology, technological shifts.

Wrap up

- The effect of network perturbations on centrality measures.
- Underlying dynamics are linear (averaging).

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Future directions: the effect of perturbations on more complex systems.

- flow dynamics in infrastructure networks;
- game theoretic models in financial and economic networks;
- opinion formation and evolution in social networks.